Fixed Parameter Algorithms for Completion Problems on Plane Graphs

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in collaboration with

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The SUBGRAPH ISOMORPHISM PROBLEM (S.I.) and the MINOR ISOMORPHISM PROBLEM (M.I.) (also known as MINOR CONTAINMENT) are two well-known NP-complete problems that accept as input two graphs G and H and check whether G has any subgraph or minor isomorphic to H.

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S.I.	?	$2^{O(k)} \cdot n$ (Eppstein 1999)
M.I.	$g(k) \cdot n^3$ (Robertson & Seymour 1995)	$O(2^{O(k)} \cdot n + n^2 \cdot \log n)$ (Adler et al. 2010)

where $n = |V(G)|$ and $k = |V(H)|$.

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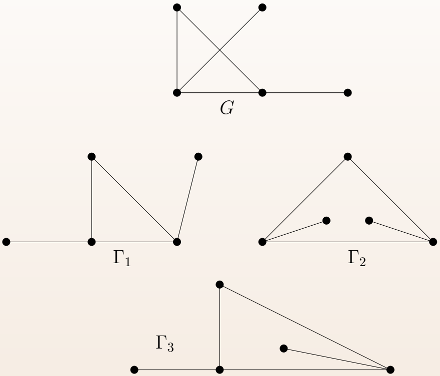
For example:



Here, G is a planar graph and Γ_1, Γ_2, and Γ_3 are planar embeddings of G. In fact, Γ_1 and Γ_2 are equivalent (topologically isomorphic) to each other but not to Γ_3.

Completion Problems

Problem: Π
Input: Graphs $G_1, \ldots, G_l$
Question: Do the graphs have a specified property P ?

Problem: Π-Completion
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Many interesting problems, naturally parameterized by the number of new edges (k), arose with the introduction of the completion operation, which have been studied a lot lately.

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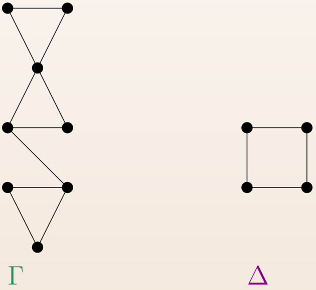
The Plane Subgraph Completion Problem

Plane Subgraph Completion (PSC)

Input: A "host" plane graph Γ and a "pattern" **connected** plane graph Δ.

Parameter: $k = |V(\Delta)|$

Question: Can we add edges to Γ so that it contains a subgraph topologically isomorphic to Δ **while remaining planar**?



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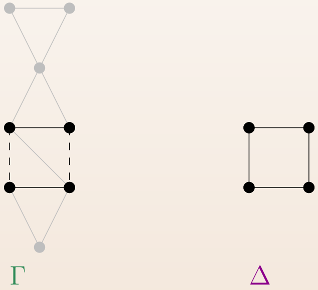

Plane Topological Minor Completion (PTMC)

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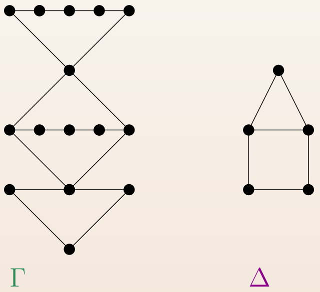

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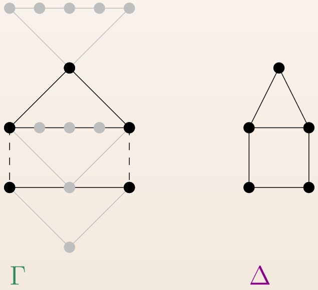

Our Results

If $k := |V(\Delta)|$ and $n := |V(\Gamma)|$, we give:

- an FPT algorithm for PSC that runs in time $2^{O(k \log k)} \cdot n^2$ and
- an FPT algorithm for PTMC that runs in time $g(k) \cdot n^2$.

Remark. In fact we can even solve more general problems: we can ask that the pattern graph Δ be given as a **planar** graph and check whether **any** of its embeddings can be found in the host.

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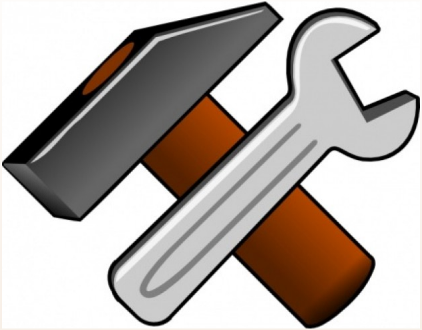
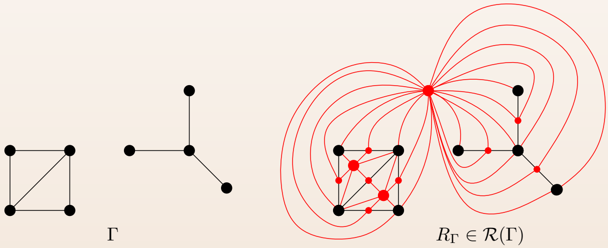


First, let's see the tools we need for the PSC-algorithm...

A *subdivided radial enhancement* of a plane graph Γ is a **plane multigraph** R_Γ, that can be constructed from Γ by subdividing each edge of the graph once and then adding a vertex inside each face and connecting it with all the vertices of the face, so that in the resulting graph embedding each face with at least one original edge is a triangle.

Example:



From now on we will call this construction just *enhancement*.

Some Observations

Let's consider some facts about this construction.

- If Γ is **disconnected**, then the enhancement is connected but it can be done in (exponentially) many ways.
- If Γ is **connected**, then the enhancement is uniquely defined (and in fact 2-connected).
- If Γ is **2-connected**, then the enhancement is 3-connected.

Whitney's Theorem (1932): Any 3-connected planar graph admits a unique embedding on the plane (up to topological isomorphism).

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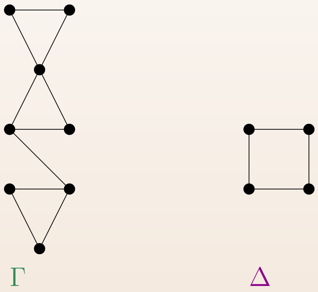
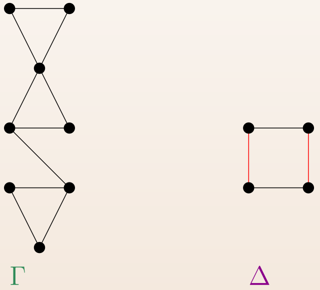


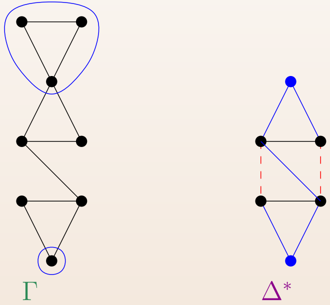
Step 1: Guess which edges of Δ (red) are missing from Γ. This is much easier than guessing which edges should be added to Γ.

$O(2^k)$ time



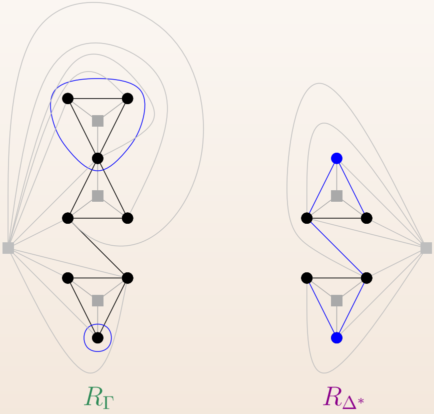
Step 2: Guess a supergraph Δ^* of Δ with extra (blue) vertices and edges in some faces that represent vertices and edges of Γ inside the corresponding faces. Then remove the red edges.

$O(2^{k \log k})$ time



Step 3: Enhance Γ arbitrarily and "guess" an enhancement of Δ^*, resulting in R_Γ and R_{Δ^*} respectively.

$O(n + 2^k)$ time



Step 4: Enhance twice more both of the graphs. This is to ensure that both of the resulting graphs $Q(\Gamma)$ and $Q(\Delta)$ are 3-connected and therefore, due to Whitney's theorem, uniquely embeddable.

$O(n+k)$ time

Step 5: Pick a vertex u of Γ and contract everything in $Q(\Gamma)$ that is at a distance greater than $\mathbf{diam}(Q(\Delta)) = O(k)$ from u. It is easy to prove that the resulting graph $Q_u(\Gamma)$ has treewidth $\leq 3 \cdot \mathbf{diam}(Q(\Delta)) = O(k)$.

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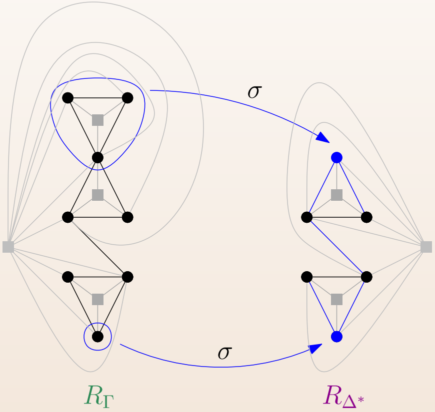
$O(n)$ time

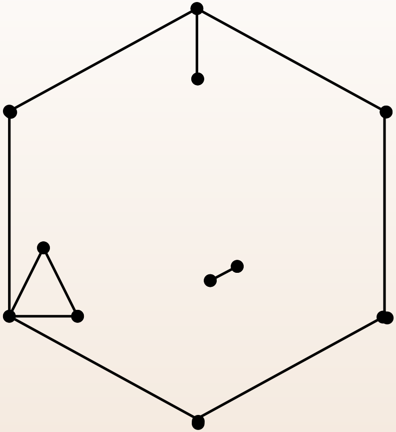
Step 6: Use a modified algorithm by Adler et al. (2011) to check whether **the planar graph** $Q_u(\Gamma)$ contains **the planar graph** $Q(\Delta)$ as a minor. This is easy since both of the graphs have now size $O(k)$. If the algorithm answered "NO", go back to step 5 and pick a different vertex.

$\leq n$ steps



More tools are needed for the PTMC-algorithm...

Cylindrical Enhancement



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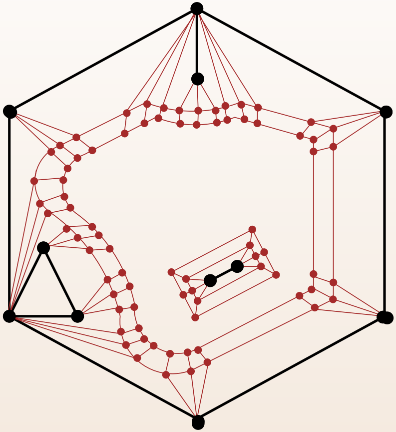

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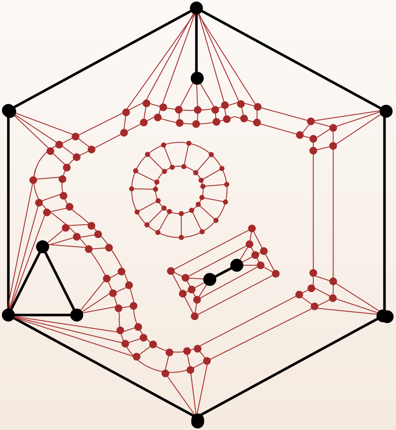

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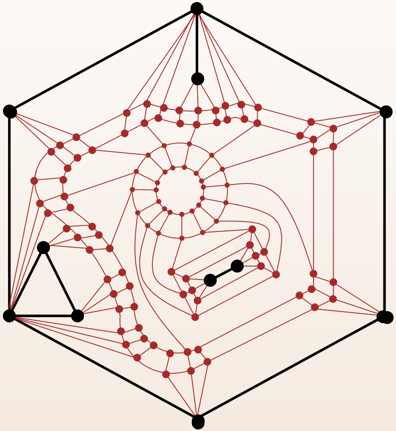

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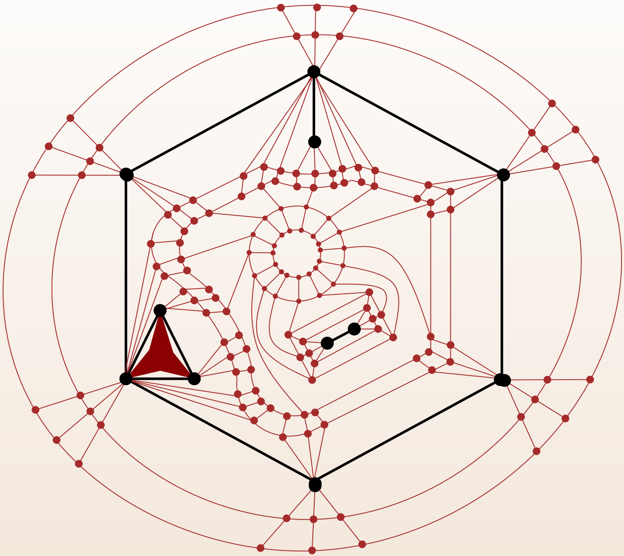
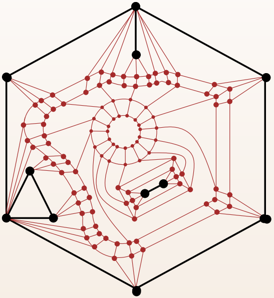


The resulting graph Γ^c has $O(n)$ vertices.



- We have proved that Δ is a completion-topological-minor of Γ **iff** Δ is a special-topological-minor of Γ^c, where the vertices of Δ are associated only to original vertices of Γ.
- This special relation ($\leq^*$) can be expressed in MSOL to find a top. minor that is **isomorphic** (not topologically isomorphic) to Δ.

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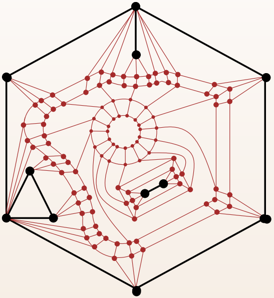

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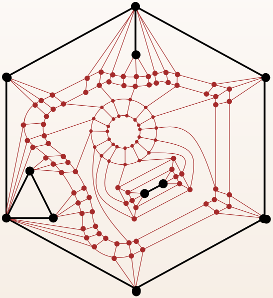

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Rooted Disjoint Paths

- To prove the previous claim, we use a result by Adler et al. (2011) which states that the number of edges that need to be added in each face in order to find k disjoint paths is bounded by $f(k)$.

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The Irrelevant-Edge Algorithm

We combine two known algorithms in order to find an irrelevant edge in the graph (i.e., an edge whose removal results in an equivalent instance) in time $g(k) \cdot n$:

- by Golovach,Kamiński,Maniatis,Thilikos (2015), we find a large wall with some special properties in the graph and
- by Kaminski,Thilikos (2012), we find an irrelevant edge in the wall.

The PSC-Algorithm

Step 1: Cylindrically enhance Γ into Γ^c.

$O(n)$ time

Step 2: If $\mathbf{tw}(\Gamma^c) \leq f(k)$, proceed to step 3. Otherwise, find an irrelevant edge in Γ^c and remove it. Repeat this step until the treewidth of the resulting graph Γ^{c-} is $\leq f(k)$.

$\leq g(k) \cdot n^2$ time

Step 3: Enhance twice Γ^{c-} and Δ, resulting in $\tilde{\Gamma}$ and $\tilde{\Delta}$.

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Step 4: Use Courcelle's algorithm to check whether $\tilde{\Delta} \leq^* \tilde{\Gamma}$.

$\leq h(k) \cdot n$ time

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Side-Results / Future work

- We can modify the PSC-algorithm to check if the pattern graph appears as **induced subgraph** in the host.
- Although the PTMC-algorithm works for **minors** as is, we can modify it slightly to obtain a linear algorithm (w.r.t. n).
- Try to drop the super-exponential factor $2^{O(k \log k)}$ of PSC to just exponential. A better way to "guess" the blue parts in the pattern will be needed.

Thank you!